HYPER VERSION

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T-duality can fail

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Abstract: We show that T-duality can be broken by non-perturbative effects in string coupling. The T-duality in question is that of the 2-torus when the heterotic string is compactified on K3$\times T^2$. This case is compared carefully to a situation where T-duality appears to work. A holonomy argument is presented to show that T-dualities (and general U-dualities) should only be expected for large amounts of supersymmetry. This breaking of $R \leftrightarrow 1/R$ symmetry raises some interesting questions in string theory which we discuss. Finally we discuss how the classical modular group of a 2-torus appears to be broken too.

Keywords: Superstrings, Heterotic Strings, String Duality Supersymmetry and Duality.

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Contents

1. Introduction

$R \leftrightarrow 1/R$ symmetry, the statement that a string theory compactified on a circle of radius R is isomorphic to a string compactified on a circle of radius $1/R$, is the most basic instance of T-duality. It is not only a prime example of inherently "stringy" physics but also an extremely useful tool in studying the theory. It has been used to argue for the existence of "minimal distances" in string theory and, applied in more complicated situations, to derive various properties of the theory. In these applications it is often necessary to assume that T-duality is an exact symmetry of string theory. The justification of $R \leftrightarrow 1/R$ symmetry comes from conformal field theory. Probably the neatest argument is that if one looks at a conformal field theory corresponding to a string on a circle of radius R, then at $R = 1$ (in suitable units) one gets an extra SU(2) symmetry which may be used to identify the marginal operator which decreases R with the marginal operator which increases R (see, e.g., [1]). In compactification on T^2 the T-duality is extended to an $\mathrm{SL}(2, \mathbb{Z})$ symmetry acting on the volume and background B-field.

The problem with the argument is that it ignores non-perturbative effects in the string coupling constant. In this paper we will show how non-perturbative effects can modify the conclusion when the string coupling is nonzero. Our general conclusion is that T-duality-like arguments work generally only when there is a lot of supersymmetry. Here we will look at the heterotic string compactified on a product of a K3 surface and a 2-torus leading to an $N = 2$ theory in four dimensions. Heterotic strings compactified on T^2 exhibit T-duality at any value of the string coupling, as we will show. At the level of conformal field theory the correlation functions factorize into products of correlators associated to one or the other factor of the compactification space, so the presence of K3 is irrelevant to the issue of T-duality for T^2. We will show that this is not true nonperturbatively. The 2-torus in this context appears to be the most supersymmetric case for a failure of T-duality and so will be the easiest to analyze.

In many ways the essential facts of this paper have been known to many people for a few years now. That is, something peculiar happens to T-duality when a heterotic string is compactified on K3$\times T^2$. This was studied in particular in [2, 3, 4] for example. Our purpose here is to emphasize that this implies a basic failure of T-duality in a general setting — a fact which does not appear to have been widely appreciated.

We should immediately clarify precisely what we mean when we say that T-duality is "broken". What we do *not* mean is that we consider the string theory on a circle of radius R and then the same string theory on a circle of radius $1/R$ and discover that they are different. The situation is more awkward than this because it is difficult to say unambiguously how to measure the radius of a circle in this context.

What we can describe unambiguously is the moduli space of heterotic string theories compactified on a given spacetime. We can then follow the philosophy developed in studying moduli spaces of Calabi-Yau threefolds [5, 6, 7]. We attempt to assign a "geometrical" interpretation in terms of parameters such as the sizes of circles to points in the moduli space. One begins with some large radius (and/or weak coupling) region where one understands the system classically and then one integrates along paths in the moduli space assigning parameters to all the points in the moduli space. The problem with this method is that monodromies within the moduli space force branch cuts to be made. In general one finds that this process requires choices, so that the parameters are not uniquely determined by the moduli. One may try to remove these monodromies by forming a Teichmüller space which will cover this moduli space. If all goes well then the new copies of the fundamental region will tessellate "naturally" to form a cover giving some parameter space. An example where this fails was first recognized for measuring the volume of the quintic threefold in [5] (see figure 5.2 of that paper in particular).

It is important to note that one may always form the Teichmüller space and obtain an associated (usually very large) modular group. Indeed this was the approach taken in [2] for example and one may also obtain useful information about the theory if this path is taken (see [4] for example). The point we wish to emphasize here however is that this Teichmüller space is often physically meaningless in the usual sense.

What conditions need to hold in order for a physical interpretation to survive? In this paper we propose the following. The parameter space must contain a limiting "semiclassical" open region corresponding to weakly coupled strings and large radii. This region should be unique up to the action of duality transformations which can be determined from the semiclassical approximation to the system. In the case at hand this would mean that the large-radius, weak-coupling limit of a T^2 compactification must be unique up to $B \to B+1$. This requirement is quite restrictive as we shall see.

We believe this is a very reasonable definition for T-duality to respect. One could not seriously propose that there is a symmetry in the universe which identifies a circle of radius 1cm with a circle of radius 2cm — we may use a ruler to measure the difference! An important aspect of T-dualities is that large distances are unique. It is precisely this uniqueness which is lost if we allow an arbitrary cover of the moduli space to act as our Teichmüller space.

We will endeavour to show that the moduli space for the 2-torus in the context above does not admit a covering with this property and so there is no T-duality group. Note that $R \leftrightarrow 1/R$ symmetry is not literally "broken". Instead what is true is that for R of order one or smaller, there is no natural way to associate a "size" R to a given point in moduli space. In particular, the statement of T-duality thus loses its content.

The non-perturbative corrections responsible for modifying the structure of the moduli space and rendering it inconsistent with T-duality are presumably represented by fivebrane instantons wrapping the K3 as well as one of the cycles of the torus. We do not know at present how to compute these corrections explicitly. Instead, we use a dual type IIA description of the model. In this description, the heterotic dilaton is mapped to a Kähler modulus and the corrections in question are generated by world-sheet instantons whose effect is readily computed by mirror symmetry.

In the context of this dual model the issue of T-duality breaking becomes that of *fibre-wise duality*. The idea is that one has a Calabi-Yau space, X, which is an F-fibration for some fibre F which is also a Calabi-Yau space. If some duality statement is true for each F can it be extended to a duality of X? In section 2 we give an example where it can and in section 3 we give an example where it cannot. By heterotic/type II string duality the latter case implies a failure of T-duality. The essential difference between sections 2 and 3 is that in former we show that we are required to fit a *finite* number of fundamental domains of the moduli space into a part of the Teichmüller space with *infinite* area while in the later case we would be required to fit an *infinite* number of fundamental domains of the moduli space into a part of the Teichmüller space with *finite* area.

In section 4 we will discuss the consequences of broken dualities. In particular we will set a general description of how dualities are natural only when there is a large amount of supersymmetry. We propose that the clearest interpretation of the results in this paper is that *whether a circle respects T-duality in string theory depends on the context of the circle.* We will also note the possibly alarming conclusion that the *classical* $\mathrm{SL}(2,\mathbb{Z})$ modular group of the moduli space of complex structures on a torus is also probably broken (in the same sense).

2. Unbroken T-duality

2.1 A two parameter K3 surface

In this section we will deal with a type IIA string compactified on a K3 surface which is an elliptic fibration with section. We wish to see that the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the elliptic fibre can be seen in the moduli space of the K3 surface.

An F-theory argument shows that this is equivalent to saying that the $\mathrm{SL}(2,\mathbb{Z})$ T-duality acting on the moduli space of complexified Kähler forms is unaffected by the value of the complex structure modulus (or, equally, the mirror of this statement). Note that the heterotic dilaton in eight dimensions lives in a separate $\mathbb{R}$ factor of the moduli space and will certainly not affect any T-duality statements.

We know from Narain [8] that the moduli space required for a heterotic string on T^2 (with the Wilson lines switched off) is

$$\mathcal{M}_0 = \frac{\mathrm{O}(\Gamma_{2,2})\backslash\,\mathrm{O}(2,2)}{\big(\mathrm{O}(2)\times\mathrm{O}(2)\big)}\,, \tag{2.1}$$

where $\Gamma_{2,2} = U \oplus U$ and U is the usual hyperbolic lattice of signature $(-1,1)$. This space can be thought of as the grassmannian of space-like 2-planes in $\mathbb{R}^{2,2}$ where we care only about the orientation of the 2-planes relative to the lattice $\Gamma_{2,2}$.

One may explicitly map $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{O}(2,2)$ by writing a vector (x_1,x_2,x_3,x_4) in $\mathbb{R}^{2,2}$ as

$$M = \begin{pmatrix} x_1 & x_3 \\ x_4 & x_2 \end{pmatrix}, \tag{2.2}$$

and then let $(A,B)\in\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ act on this vector as

$$(A,B) : M \longrightarrow AMB^{-1}\,. \tag{2.3}$$

(Note that $(-1,-1)$ maps to the identity.)

This allows us to rewrite $\mathcal{M}_0$ in (2.1) in terms of two copies of the upper-half plane $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$. The modular group $\mathrm{O}(\Gamma_{2,2})$ then generates two copies of $\mathrm{SL}(2,\mathbb{Z})$ — each acting on its upper-half plane, a "mirror" $\mathbb{Z}_2$ which exchanges the upper-half planes, and a "conjugation" $\mathbb{Z}_2$ which acts as (minus) complex conjugation on both upper-half planes simultaneously. One can then interpret one upper-half plane as representing the complex structure, τ, of the 2-torus and the other upper-half plane as representing the B-field and Kähler form, $\sigma = B + iJ$, of the torus [9]. Equivalently we may describe this moduli space as two copies of the j-line divided by exchange and complex conjugation. We denote coordinates on the two j-lines by j_1 and j_2.

Via the usual F-theory argument [10] the heterotic string on T^2 is dual to F-theory (or the type IIA string in some limit) on a K3 surface, S.[1] We will map the moduli space explicitly for the case of an algebraic K3 surface written as a hypersurface in a toric variety. Let us consider the surface $\tilde{S}$ given by

$$x_0^2 + x_1^3 + x_2^{12} + x_3^{12} + \psi\, x_0\, x_1\, x_2\, x_3 \;+ \phi\, x_2^6\, x_3^6\,, \tag{2.4}$$

in $\mathbb{P}^3_{\{6,4,1,1\}}/\mathbb{Z}_6$, where we divide by the $\mathbb{Z}_6$ action generated by $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, e^{\pi i/6}x_2, e^{-\pi i/6}x_3)$.[2] This K3 surface is mirror via the usual arguments [11, 12] to an elliptic K3 surface with a section. That is, S and $\tilde{S}$ are a mirror pair in the sense of algebraic K3 surfaces [12, 13, 14]. We should now be able to map out the desired moduli space, $\mathcal{M}_0$ by varying ψ and ϕ in (2.4), i.e., by varying the complex structure of $\tilde{S}$.

We want to explicitly map the coordinates (ψ, ϕ) to the two copies of the j-line we saw above. This has already been done in [15, 16] (see also [17]). Let us review this construction so that we may compare some details to that of the Calabi-Yau threefold in the next section. The structure is most easily seen by finding the "interesting" points in the moduli space. As is well known, at certain points in the moduli space we obtain enhanced gauge symmetries. In the language of the grassmannian (2.1) this occurs when lattice elements of length squared -2 are orthogonal to the space-like 2-plane. Let us suppose that we force the 2-plane to be orthogonal to such a lattice element. The result is that the 2-plane now varies in the space $\Gamma_{2,1} \otimes_{\mathbb{Z}} \mathbb{R}$, where $\Gamma_{2,1} = \mathrm{U} \oplus \mathrm{L}(2)$ and $\mathrm{L}(2)$ is a one-dimensional lattice whose generator has length squared 2. That is, the moduli space within $\mathcal{M}_0$ where the theory has a gauge symmetry of at least SU(2) is given by

$$\mathcal{M}_{\mathrm{SU}(2)} = \frac{\mathrm{O}(\Gamma_{2,1})\backslash\, \mathrm{O}(2,1)}{(\mathrm{O}(2) \times \mathrm{O}(1))}\,. \tag{2.5}$$

In the same way that $\mathrm{O}(2,2)$ is mapped to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, we may map $\mathrm{O}(2,1)$ to $\mathrm{SL}(2,\mathbb{R})$ by writing

$$M = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}, \tag{2.6}$$

and letting $A \in \mathrm{SL}(2,\mathbb{R})$ act as

$$A : M \to AMA^{-1}\,. \tag{2.7}$$

[1] There is a very awkward $\mathbb{Z}_2$ identification problem which we will ignore for the sake of exposition. The complex conjugation symmetry of $\mathcal{M}_0$ must be treated carefully in the type IIA picture. We will ignore this subtlety for the sake of exposition.

[2] Although this action looks like it generates $\mathbb{Z}_{12}$, don't forget that the weighted projective identification means that $(x_0, x_1, x_2, x_3) \cong (x_0, x_1, -x_2, -x_3)$.

The reader may also check that (up to a $\mathbb{Z}_2$ corresponding to complex conjugation) $\mathrm{O}(\Gamma_{2,1}) \cong \mathrm{SL}(2,\mathbb{Z})$.

Thus we may embed $\mathcal{M}_{\mathrm{SU}(2)}$ into $\mathcal{M}_0$ by setting $A = B$. In other words $j_1 = j_2$ in terms of the two j-lines. Actually the fact that $\mathrm{O}(\Gamma_{2,2})$ acts *transitively* on vectors of length squared -2 shows that we have an enhanced gauge symmetry (of at least SU(2)) if *and only if* $j_1 = j_2$.

In terms of $\tilde{S}$ in the form (2.4) we have an enhanced gauge symmetry when ϕ and ψ are tuned to produce a canonical singularity. This happens when the discriminant, Δ, of the equation vanishes. This is given by[3]

$$\Delta = (\phi - 2)(\phi + 2)\big(432\phi - \psi^6 + 864\big)\big(432\phi - \psi^6 - 864\big)\,. \tag{2.8}$$

One may also argue that $\Delta = k(j_1 - j_2)^2$, where k is some constant, as follows. As we said above we only get enhanced gauge symmetry when $j_1 = j_2$. One argues that the power is 2 by saying that any odd power would violate the $j_1 \leftrightarrow j_2$ symmetry and any power higher than 2 would imply that the degeneration of $\tilde{S}$ when $\Delta = 0$ could never be suitably generic.

We also have further enhanced gauge symmetry. Namely there is a single point $j_1 = j_2 = 0$ where we have SU(3) and a single point $j_1 = j_2 = 1728$ where we have $\mathrm{SU}(2) \times \mathrm{SU}(2)$. With a bit of algebra we may find the corresponding values for ϕ and ψ. All said, our equations are satisfied with $k = 1$ by making j_1 and j_2 the roots of

$$j^2 - \big(\phi\psi^6 - 432\,\phi^2 + 1728\big)\,j + \psi^{12} = 0\,. \tag{2.9}$$

If the reader is not fully convinced of this map between (ψ, ϕ) and the j-lines, one may check that the "flatness" condition is satisfied for the "special coordinates" σ and τ. We will do this next. For a nice account of what is meant by "special coordinates" we refer to [18].

The special coordinates are of particular interest to us because these specify "length" are should parameterize any natural Teichmüller space. Let us perform the usual analysis of converting "algebraic coordinates" (ψ, ϕ) into special, or flat coordinates as in [5, 6, 19]. This is a rather technical process but it is thoroughly treated in the literature. Our approach is closest to [7]. Introducing variables

$$z = \frac{\phi}{\psi^6}\,, \qquad y = \frac{1}{\phi^2}\,, \tag{2.10}$$

[3]Note that there are some identifications to be made within the (ψ, ϕ) plane. This discriminant does not really have four components.

we obtain the hypergeometric partial differential operators

$$\square_z = z\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial z} - 2y\frac{\partial}{\partial y}\right) - 432\,z\left(z\frac{\partial}{\partial z} + \frac{1}{6}\right)\left(z\frac{\partial}{\partial z} + \frac{5}{6}\right),$$
$$\square_y = \left(y\frac{\partial}{\partial y}\right)^2 - y\left(z\frac{\partial}{\partial z} - 2y\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial z} - 2y\frac{\partial}{\partial y} - 1\right). \tag{2.11}$$

The monomial-divisor mirror map of [20] tells us that S is an elliptic fibration with a section and that the section goes to infinite size as $y \to 0$ and the fibre goes to infinite size as $z \to 0$. There is a unique solution of $\square_z\Phi(z,y) = \square_y\Phi(z,y) = 0$ which is equal to 1 at the the origin. Call this Φ_0. We also have solutions Φ_z Φ_y which behave asymptotically as $\log(z)$ and $\log(y)$ as we approach the origin. The flat coordinates are then ratios of these solutions in the usual way. Let us put

$$q_z = \exp\left(\frac{\Phi_z}{\Phi_0}\right), \qquad q_y = \exp\left(\frac{\Phi_y}{\Phi_0}\right). \tag{2.12}$$

(Just as q is used conventionally to denote $\exp(2\pi i\tau)$.) We then have a power series solution from the hypergeometric system

$$q_z = z - zy + 312\,z^2 - zy^2 - 192\,z^2y + 107604\,z^3 + \cdots,$$
$$q_y = y + 120\,zy + 2y^2 + 41580\,yz^2 + 5\,y^3 + \cdots. \tag{2.13}$$

Explicit computation then shows order by order that this is consistent with (2.9) with

$$j_1 = j(q_z), \qquad j_2 = j(q_zq_y), \tag{2.14}$$

where $j(q)$ is the usual j-function expansion given by $q^{-1} + 744 + 196884q + \mathrm{O}(q^2)$. This verifies that (2.9) is correct. See [15] for an explicit proof of this.

2.2 Fibre-wise T-duality

Now we wish to analyze the same two parameter model from the point of view of the elliptic fibration of the K3 surface S. We want to know if $\mathrm{SL}(2,\mathbb{Z})$ modular group of the elliptic fibre can be seen in the moduli space of S. First let us take the area of section to be huge, $y \to 0$. In this limit, the moduli properties of S are basically that of its fibre. Putting $y = 0$ we obtain

$$j_1 = j(q_z) = \frac{1}{z(1-432z)}, \tag{2.15}$$

and $j_2 = \infty$. It is very important to note that varying z gives us a *double* cover of the j_1-line. In particular, if we plot the shape of fundamental region in terms of flat coordinates (as discussed in [5, 6, 7]) we obtain the shaded area of figure 1. This region is $H/\Gamma_0(2)$, where H is the upper-half plane. This means that there is a non-toric $\mathbb{Z}_2$ symmetry acting on $\tilde{S}$ which will identify the two fundamental regions of $H/\,\mathrm{SL}(2,\mathbb{Z})$ in figure 1.

In terms of the mirror $\tilde{S}$, taking $y \to 0$ takes $\phi \to \infty$. $\tilde{S}$ is an elliptic K3 surface with a section and in this limit the complex structure of the fibre becomes constant. The remaining variation of complex structure of $\tilde{S}$ is therefore the variation of the complex structure of this constant fibre. A similar story was discussed in [21].

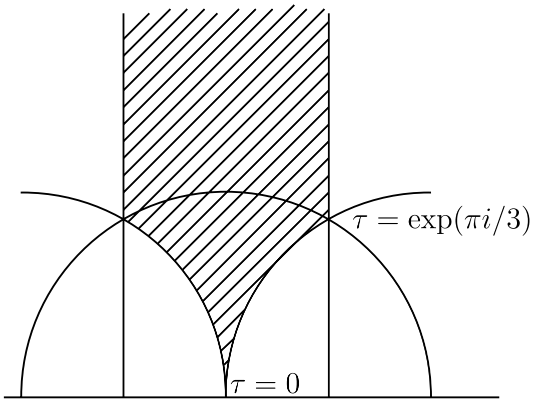

Figure 1: The domain of mapping into flat coordinates when $y = 0$.

Finding the moduli space of an elliptic curve via toric varieties is always going to be plagued by finding multiple covers of the j-line because of non-toric symmetries. Most simple families of elliptic curves that one tries to write down tend to have fixed rational points. See [22] for a discussion of such problems. This "level structure" forces the naïve moduli space to be a multiple cover of the j-line. This helps us out in the toric case as we now explain.

In the general approach of Candelas et al. [5] in analyzing a one parameter case, the moduli space will always look like a rational curve with 3 special points. To use the language of the quintic threefold, one point will be "large complex structure" another will be an orbifold-like point and a third, which marks the division between the first two phases, is a "conifold"-like point. The problem is that an elliptic curve doesn't have a conifold-like point — any singular complex structure must be the large complex structure. Thus one must have a non-toric symmetry identifying the putative conifold point with the large complex structure point. In our example this must identify the "conifold" $\tau = 0$ with $\tau = i\infty$, i.e. it is the $\tau \to -1/\tau$ symmetry. It is this non-toric discrete symmetry which produces the multiple cover of the j-line above.

Now the main question we wish to ask is what happens to figure 1 if we allow y to be nonzero? From (2.15) the point given by $\tau = 0$ in figure 1 corresponds to $z = 1/432$. This is on the discriminant as it corresponds to $j_1 = j_2 = \infty$. For general y, the relevant part of the discriminant becomes

$$(432\, z - 1)^2 - 864^2\, yz^2 = 0\,. \tag{2.16}$$

Thus, as y is switched on, the point $z = 1/432$ splits into two nearby points (up to extra identifications). How can we understand this in terms of flat coordinates?

The question of what a slice of constant y in the moduli space corresponds to in terms of the elliptic curve is awkward. Recall that setting $y = 0$ put $j_2 = \infty$ and so $\sigma = i\infty$. Suppose instead we fix σ to be iC for some large but finite C. What does the fundamental region for τ then look like? Because of the mirror map $\sigma \leftrightarrow \tau$, we are free to fix $\mathrm{Im}(\tau) \leq \mathrm{Im}(\sigma)$. This "chops off" the top of the fundamental region. This will turn figure 1 into figure 2.

Now the domain for y constant, as opposed to j_2 constant, won't quite look like figure 2 but the coarse structure should be about right. Thus we see how the point at the end of the cusp at $\tau = 0$ in figure 1 gets split and moves up a little in figure 2. (We should also point out that these supposed two points are actually identified by the $\mathrm{SL}(2,\mathbb{Z})$ action in figure 2.) When y is nonzero, a solution of z satisfying (2.16) gives a $\mathbb{Z}_2$-orbifold point on S. Thus this corresponds to an $\mathrm{SU}(2)$ gauge symmetry (in the eight-dimensional theory in question) and this point in the moduli space is a $\mathbb{Z}_2$ orbifold point.

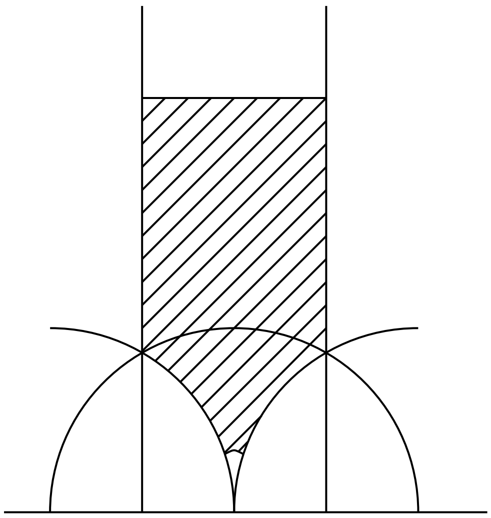

Figure 2: The domain when $\sigma = iC$.

The important point is that *nothing catastrophic happens to the* $\mathrm{SL}(2,\mathbb{Z})$ *action as we switch on* y. We just pick up an extra $\mathbb{Z}_2$ into the modular group. We know that our moduli space is of the form of a global quotient (2.1) and so no matter what kind of complex slice of the moduli space we take, we will always see some T-duality group acting nicely on some slice of the Teichmüller space. In the heterotic string point of view, the T-duality group of the torus persists.

3. Broken T-duality

We now wish to discuss along parallel lines the case of a heterotic string compactified on $\mathrm{K3}\times T^2$. To this end, we need to replace the K3 surface S with a Calabi-Yau threefold X. The space X is now a K3-fibration. The fibre has a T-duality which can be shown to be $\mathrm{SL}(2,\mathbb{Z})$ (which up to $\mathbb{Z}_2$ factors is $\mathrm{O}(\Gamma_{1,2}) \subset \mathrm{O}(\Gamma_{4,20})$ — the full T-duality group of a K3 surface). This $\mathrm{SL}(2,\mathbb{Z})$ will not survive to the full Calabi-Yau threefold moduli space. This shows that the T-duality of the heterotic string is destroyed at non-zero string coupling.

The toric data for X will be very similar to that of S, and X will again have two deformations of the B-field and Kähler form. Let $\tilde{X}$ be given by

$$x_0^2 + x_1^6 + x_2^6 + x_3^{12} + x_4^{12} + \psi\, x_0\, x_1\, x_2\, x_3\, x_4 + \phi\, x_3^6\, x_4^6\,, \tag{3.1}$$

in $\mathbb{P}^3_{\{6,2,2,1,1\}}/G$, where we divide by the $G = \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_2$ action generated by

$$\begin{aligned}
\big(x_0, x_1, x_2, x_3, x_4\big) &\longmapsto \Big(x_0, x_1, x_2, e^{\pi i/6}x_3, e^{-\pi i/6}x_4\Big)\,,\\
\big(x_0, x_1, x_2, x_3, x_4\big) &\longmapsto \Big(x_0, x_1, e^{\pi i/3}x_2, x_3, e^{-\pi i/3}x_4\Big)\,,\\
\big(x_0, x_1, x_2, x_3, x_4\big) &\longmapsto \Big(x_0, e^{\pi i/3}x_1, x_2, x_3, e^{-\pi i/3}x_4\Big)\,.
\end{aligned} \tag{3.2}$$

This Calabi-Yau threefold and its mirror X have been studied extensively [23]. In particular Kachru and Vafa [24] conjectured that the type IIA string on X is dual to the heterotic string on $S_H \times E_H$ where S_H is a K3 surface and E_H is an elliptic curve. The string theory on E_H is frozen along $\tau = \sigma$ by Higgsing the enhanced SU(2) symmetry we saw in the last section. Thus the moduli space of the string on E_H should be just a single copy of the j-line. There is considerable evidence to support this conjecture (for example [24, 25, 26]).

Now as before we may introduce

$$z = \frac{\phi}{\psi^6}\,, \qquad y = \frac{1}{\phi^2}\,, \tag{3.3}$$

and we obtain the hypergeometric partial differential operators

$$\begin{aligned}\Box_z &= \left(z\frac{\partial}{\partial z}\right)^2\left(z\frac{\partial}{\partial z} - 2y\frac{\partial}{\partial y}\right) - 1728\,z\left(z\frac{\partial}{\partial z}+\frac{1}{6}\right)\left(z\frac{\partial}{\partial z}+\frac{1}{2}\right)\left(z\frac{\partial}{\partial z}+\frac{5}{6}\right),\\ \Box_y &= \left(y\frac{\partial}{\partial y}\right)^2 - y\left(z\frac{\partial}{\partial z} - 2y\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial z} - 2y\frac{\partial}{\partial y} - 1\right).\end{aligned} \tag{3.4}$$

Again the monomial-divisor mirror map of [20] tells us that X is a K3 fibration with a section and that the section goes to infinite size as $y \to 0$ and the K3 fibre goes to infinite size as $z \to 0$. Note that this hypergeometric system is remarkably similar to the elliptic fibration in the previous section (2.11).

We may find the flat coordinates as before, the exponentials of which we will call q_z and q_y again. Now when we put $y = 0$ one finds [23]

$$j(q_z) = \frac{1}{z}\,. \tag{3.5}$$

That is, the moduli space of the K3 fibre gives one cover of the j-line. The fact that the j-line appears here can be argued directly as follows. With a suitable change of coordinates, the constant K3 fibre of (3.1) as $y \to 0$ may be written as

$$x_0^2 + x_1^6 + x_2^6 + x_3^6 + \xi\, x_2^2\, x_3^2\, x_4^2\,, \tag{3.6}$$

in $\mathbb{P}^3_{\{3,1,1,1\}}$ divided by $\mathbb{Z}_2 \times \mathbb{Z}_6$. This is manifestly a double cover of $\mathbb{P}^2$ branched over a sextic curve. We may write $\mathbb{P}^2$ as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of $\mathbb{P}^2$ by mapping $[y_1, y_2, y_3] \to [x_1^2, x_2^2, x_3^2]$. This $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of the $\mathbb{Z}_2 \times \mathbb{Z}_6$ by which we are going to orbifold. That is, the sextic curve is mapped to a cubic — i.e., an elliptic curve. Thus the moduli space of the K3 surface can be mapped to the moduli space of an elliptic curve. A little work is involved in showing that there is no level structure implicit on this elliptic curve and so one indeed gets one copy of the j-line as desired.

The heterotic interpretation of the flat coordinates corresponding to q_z and q_y are the single complex modulus of the torus and the dilaton-axion respectively [24].

JHEP08(1999)001

Indeed, the appearance of the j-line above as $y \to 0$ shows that, in this limit we do indeed get the correct moduli space for the torus — $H/\mathrm{SL}(2,\mathbb{Z})$. In this way we see that the T-duality of the heterotic string on a torus at zero coupling is reproduced in the type-IIA dual.

Now what happens when we allow y to be nonzero? Although the hypergeometric system (3.4) is similar at first sight to that of the previous section (2.11), there is a drastic difference. Let us consider the behaviour of the discriminant locus in a constant y slice as y is near zero. As discussed above, a component near $z = 1/432$ gave us two points (before identifications are made) close to where a *cusp* of the fundamental region was in the $y = 0$ limit. This cusp corresponded to $j = \infty$.

In this case in question here, the corresponding component of the discriminant locus is given by

$$(1 - 1728\, z)^2 - (3456)^2\, z^2 y = 0\,. \tag{3.7}$$

At $y = 0$, the point $z = 1/1728$ is not a cusp on the j-line but rather the $\mathbb{Z}_2$-orbifold point at $j = 1728$. In the language of the K3 fibre of X, the K3 surface acquires a $\mathbb{Z}_2$-orbifold point here (along with $B = 0$) to give an enhanced SU(2) gauge symmetry.

Now as we turn on y we have two solutions for z. This is exactly the situation that was analyzed thoroughly by Kachru et al. [24, 25] where it was shown that the region of moduli space near $z = 1/1728$ and $y = 0$ maps to Seiberg-Witten theory for an SU(2) theory [27]. The single point $y = 0$, $z = 1/1728$ is the classical limit corresponding to an SU(2) enhanced gauge symmetry and the two points for constant $y > 0$ are the points giving the massless solitons. The important result we wish to borrow from Seiberg-Witten theory is that in flat coordinates the monodromy around either of these two points alone is *infinite*. The monodromy within the curve $y = 0$ around both points together is $\mathbb{Z}_2$ — as we see from the $y = 0$ limit.[4]

The situation here contrasted with that of the previous section can be summed up as follows

- For the moduli space of S we have a point at $y = 0$ around which the monodromy is infinite. As y is switched on this splits into two points around which the monodromy is $\mathbb{Z}_2$.

- For the moduli space of X we have a point at $y = 0$ around which the monodromy is $\mathbb{Z}_2$. As y is switched on this splits into two points around which the monodromy is infinite.

This can all be traced back to the fact that for S, $y = 0$ gave a double cover of the j-line while for X it gives a single cover.

[4]This is not to say that the monodromy around the two points in the Seiberg-Witten plane is $\mathbb{Z}_2$. See [25].

We now wish to claim that this behaviour for X makes it completely impossible to maintain any notion of T-duality when y is switched on. That is T-duality for the heterotic string is "broken" as soon as the dilaton is switched on.

Suppose we let y be small but nonzero. This should correspond to the weakly-coupled heterotic string. The moduli space for the T^2 part must look "almost" like the classical fundamental region for $H/\,\mathrm{SL}(2,\mathbb{Z})$. By "almost" we mean that no points in the moduli space for T^2 can have moved a large distance with respect to the special Kähler metric relative to the $y=0$ slice. We should now ask whether we can apply $\mathrm{SL}(2,\mathbb{Z})$ to this new moduli space to form a nice tessellated cover of the upper half plane just as we could for $y=0$. Note in particular the angle of any corner of the moduli space must correspond to the monodromy in the Levi-Civita connection of the special Kähler metric around that point in order that such a tessellation works.

In the case of X this correspondence fails. We show schematically[5] what happens as y is switched on in figure 3. Near the edge of the moduli space labelled by L we need to continue into a new fundamental region. This region must be distinct from that obtained by $\tau \to \tau+1$ (otherwise there would be no monodromy around the dots labelling the discriminant). Clearly this is impossible without moving off onto a infinitely-sheeted cover of the upper-half plane.

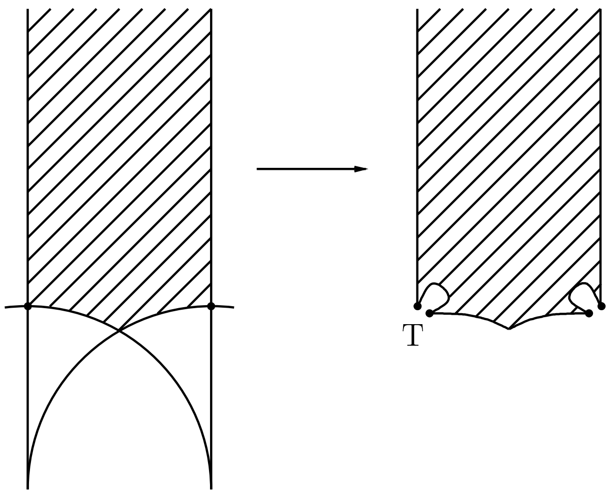

Figure 3: The moduli space for constant y for X.

Note that one might want to extend the idea of duality by following the route of building an infinite-sheet covering. Indeed this was the idea behind such duality groups as those proposed in [2] for example. One is certainly free do this mathematically and such a model might have many uses. What is unclear however is the physical significance of the huge Teichmüller space one builds by this process. It is certainly a considerable departure for the usual meaning of T-duality or U-duality. If we want to have a meaningful Teichmüller space then we should insist that we have a unique idea of an object at large radius and weak string coupling. This forces us to cover our moduli space with only one copy of a plane. We will demand this for our notion of duality.

The essential difference between this section where T-duality is broken and the previous section where T-duality was exact is as follows. In the unbroken case we

[5] We do not claim that this picture represents the geometry very accurately. The only fact we really need is that there is infinite monodromy at a point in the moduli space which is at finite distance.

introduced new copies of the fundamental region close to the horizontal axis "Im(τ) = 0" when we switched on the coupling in figure 2. In this section we had to attempt to add new regions near the point $\tau = i$. The metric diverges near $\mathrm{Im}(\tau) = 0$ allowing room for new regions. New regions cannot be fitted in near $\tau = i$ where the metric is finite.

We have also seen explicitly how the argument for T-duality using the enhanced gauge symmetry fails in this case. The Weyl symmetry of SU(2) was used to relate the deformations increasing and decreasing R from its value at the point of enhanced symmetry. In fact, the symmetry is never restored, and near the origin the monodromies about the two singularities that do appear render any attempt to consistently define the Weyl-noninvariant coordinate (a in [27]) futile.

4. Discussion

4.1 Holonomy arguments

One my try to forge a general setting for T-duality (and S-duality and U-duality) as follows. One wants to begin with a general smooth Teichmüller space, $\mathcal{T}$, which parameterizes familiar concepts such as length, coupling constants, etc. One then wants the moduli space to look like $\mathcal{M} = \mathcal{T}/G$ for some discrete duality group G. In particular this means that any orbifold point in $\mathcal{M}$ is a *global* orbifold point. That is, there is a global cover which removes this singularity.

Orbifold points in a moduli space need not be global. We would like to consider the appearance of local orbifold points as an obstruction to duality groups. Of course one may want to declare in such a case that the covering $\mathcal{T}$ itself has the orbifold point. One is free to do this and one ends up with a weaker notion of duality groups. In a typical case however this weak duality group will probably be trivial and so this is not a particularly useful notion. For duality we will insist that *all* orbifold points are global and $\mathcal{T}$ is smooth.

Thanks to the Berger-Simons theorem (see for example chapter 10 of [28]) knowing the holonomy of a manifold can lead a considerable knowledge of its structure. If the holonomy (on the Levi-Civita connection) and its representation on the tangent bundle is of a certain type then the manifold must be a symmetric space. We will call such a holonomy "rigid". If the holonomy is "nonrigid" then the manifold is of type general Riemannian, complex Kähler, hyperkähler, etc. By using the homogeneous structure of the symmetric spaces one may argue the following by tracing geodesics.[6]

Proposition 1 *If a geodesically complete orbifold has a rigid holonomy then it is globally a quotient of a symmetric space.*

[6] We thank R. Bryant for discussions on this.

Note that in lower dimensions, a sufficiently extended supersymmetry forces a rigid holonomy group. The proposition tells us immediately that we should expect duality groups to appear for a large number of supersymmetries. For example for a type IIA string on a K3 surface we have $N = 2$ supersymmetry in six dimensions giving an R symmetry of $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{SO}(4)$ (up to discrete groups). Given the dimension of the moduli space this has rigid holonomy [29] and so we see a Teichmüller space in the form of a symmetric space and a T-duality group (or more generally a U-duality group). The same is not true for an $N = 2$ theory in four dimensions however. Here the holonomy of the moduli space only dictates that it be the product of a Kähler manifold and a quaternionic Kähler manifold. This non-rigid case should not be expected to have a duality group. This is the case of a heterotic string on K3$\times T^2$.

This classification breaks down a little in higher dimensions. This happens because symmetric spaces can sometimes have a holonomy (and representation) which appears to be nonrigid. For example, the heterotic string on a 2-torus alone is an $N = 1$ theory in eight dimensions which has an R symmetry U(1). This tells us only that the moduli space is Kähler and so would not in itself predict a T-duality group. Nevertheless the moduli space here is a quotient of a symmetric space. This shows that the holonomy can only tell us when we *must* have a duality group rather than when we cannot. It seems reasonable however in lower dimensions where this ambiguity does not exist that a generic compactification leading to non-rigid holonomy will have no T-duality. The computation of section 3 shows this is the case at least for one example.

4.2 Consequences

We have shown that when one considers a heterotic string compactified on K3$\times T^2$ in a particular way then the factor of the moduli space relevant to the torus is *not* a quotient of a physically meaningful smooth space of parameters by some T-duality group. In particular it is a completely meaningless statement to say that there is some kind of $R \leftrightarrow 1/R$ symmetry on the circles within this torus. For a non-zero value of the string coupling constant the torus (and therefore circle) obeys a T-duality relationship no more than a generic Calabi-Yau threefold. The meaninglessness of T-duality for Calabi-Yau's was discussed in [5, 6, 7].

All one may do is to begin labelling points in the moduli space with size values near the large radius limit and then extend this process over the entire moduli space. This will usually lead to a minimum "size". No real meaning can be attached to sizes below this value.

The question that immediately raises itself is the following. *If T-duality fails for the heterotic string on K3$\times T^2$ then does it also fail for a circle itself?* One might argue that if the string on a circle of radius R is exactly the same thing as a string on a circle of radius $1/R$ then further compactification on K3$\times S^1$ should also yield identical physics. This leads to three possibilities

1. *T-Duality fails for the string on a circle.* That is, $R \leftrightarrow 1/R$ symmetry is totally wrong when all non-perturbative corrections are taken into account. This would seem to be unlikely. For a string on a single circle at its self-dual radius we really see an unbroken SU(2) gauge symmetry signaling an identification in the moduli space. This SU(2) is broken by quantum effects when we consider K3$\times T^2$. The holonomy arguments above also show that a lot of supersymmetry implies T-duality. We suggest therefore that T-duality is a symmetry for a string on a circle by itself.

2. *Heterotic/Type II duality is wrong.* If we really want T-duality to be respected exactly by all strings then we need to sacrifice the duality we used in this paper. One is free to do this. We do not know enough about string theory to "prove" heterotic/type II duality. If one really is wedded to the idea of T-duality then this is a sacrifice one might be willing to make.

3. *A string doesn't know about its compactifications.* When one says that one understands some string theory compactified on some space, does this understanding ever include further compactification? Such a further compactification includes putting different boundary conditions on fields which had lived in flat space. In particular one may break supersymmetry. The above reasoning that $R \leftrightarrow 1/R$ symmetry on a circle implies T-duality for further compactification is wrong.

The last possibility would appear to be the most reasonable. It implies that $R \leftrightarrow 1/R$ symmetry of a circle *depends on the context of the circle.* Any argument which uses T-duality for a circle or torus embedded in some more complicated situation cannot necessarily be considered rigorous.

In this paper we have considered the Kachru-Vafa example of the heterotic string compactified on a torus stuck at "$\sigma = \tau$". This allowed us to consider just a two-parameter family of Calabi-Yau threefolds — one remaining parameter for the torus and the string coupling. Our failure to see an $\mathrm{SL}(2,\mathbb{Z})$ modular group can therefore be viewed just as much as failure of the *classical* $\mathrm{SL}(2,\mathbb{Z})$ *modular group* as of T-duality. What one should do is to go to the three parameter example of [24] so that we may disentangle the σ and τ parameters of the 2-torus. The essential computation was again done in [25]. The coarse result of this is that the moduli space does not respect *any* $\mathrm{SL}(2,\mathbb{Z})$ symmetry generically.

This is perhaps a good deal more shocking at first sight than breaking T-duality. Are we really saying that string theory breaks the classical modular invariance of a torus? In other words string theory does not respect diffeomorphism invariance of the target space!

JHEP08(1999)001

This is unavoidable if one wants to maintain heterotic/type II duality. Actually this possibility is not as bad as it might first appear upon reflection.

One is used to the idea from conformal field theory at weak string coupling that quantum geometry may effect the structure of the moduli space of complexified Kähler form but that the complex structure moduli space is unchanged. Now we claim that string coupling effects may also lead to quantum corrections to the moduli space of complex structures. This is quite reasonable in the current context as once string corrections have been taken into account the moduli space of the 2-torus does not even factorize exactly into Kähler form and complex structure parts. Note also that the parts of the moduli space whose shape is most effected by the corrections are those close to where the enhanced gauge symmetry appeared in the weak coupling limit. These must involve circles in the torus approaching the string scale. So it is still distances at the string scale in some sense which are most affected by quantum effects.

Again we should emphasize is that we are not really saying that a string on a certain torus is different to a string on the torus having undergone a modular transform. Rather we are saying that once the string coupling is not zero, the moduli space of is no longer of the form $H/\,\mathrm{SL}(2,\mathbb{Z})$. The meaning of what is it to go outside the fundamental region then becomes unclear.

Finally we note that all of the results in this paper have tied to dualities of the heterotic string. One might wish the type II string to escape such effects. This appears to be unlikely. Our holonomy argument of section 4.1 would seem to imply that a generic compactification with little supersymmetry should not obey any T-duality or U-duality laws. What is true however is that the type IIA string has more supersymmetry to begin with. Thus we know that T-duality for the type IIA string K3$\times T^2$ is exact (see, for example, [30] for a discussion of this). What one might question however is whether the same is true for the type II string on $Z \times S^1$ where Z is a generic Calabi-Yau threefold.

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JHEP08(1999)001